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MATH 212  
Assignment 5

3.2.27 -  $f(x) = e^x$

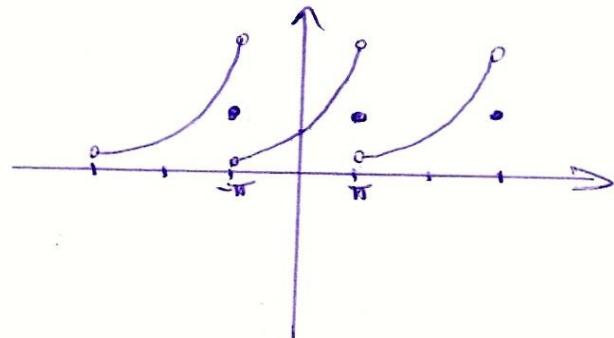
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$$\tilde{F}(f(x)) = \frac{1}{2\pi} (e^\pi - e^{-\pi}) + \sum_{k=1}^{\infty} \frac{(-1)^k (e^\pi - e^{-\pi})}{\pi(k^2 + 1)} \cos kx + \frac{(-1)^{k+1} \ln(e^\pi - e^{-\pi})}{\pi(k^2 + 1)} \sin kx$$

It converges to  $\begin{cases} e^{x-2m\pi} & (2m-1)\pi < x < (2m+1)\pi \\ \frac{e^\pi + e^{-\pi}}{2} & x = (2m+1)x \end{cases}$

$$(2m-1)\pi < x < (2m+1)\pi$$

$$x = (2m+1)x \quad m \in \mathbb{Z}$$



3.2.51 - a)  $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$  (or you can find  $c_n$  by an integration by parts similar to 3.27)

5)  $f(x) = x$

$$\tilde{F}(f(x)) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-ikx} dx$$

$$\begin{aligned} \text{If } k \neq 0: &= \frac{1}{2\pi} \left[ -\frac{x}{ik} e^{-ikx} + \frac{1}{k^2} e^{-ikx} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[ -\frac{\pi}{ik} e^{-ik\pi} + \frac{1}{k^2} e^{-ik\pi} - \frac{\pi}{ik} e^{ik\pi} - \frac{1}{k^2} e^{ik\pi} \right] \\ &= \left( \frac{i}{2\pi k} + \frac{1}{2\pi k^2} \right) e^{-ik\pi} + \left( \frac{-1}{2\pi k^2} + \frac{i}{2\pi k} \right) e^{ik\pi} \end{aligned}$$

If  $k=0$ :  $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0$

$$\tilde{F}(x) = \sum_{k=-\infty}^{+\infty} c_k e^{ikx}$$

, the expression of  $c_k$  is written above

$$d) f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

$$\tilde{FS}[f(x)] = \sum_{k=-\infty}^{\infty} c_{ka} e^{ikx}$$

$$c_{ka} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \left[ \int_{-\pi}^0 -xe^{-ikx} dx + \int_0^{\pi} xe^{-ikx} dx \right]$$

$$\underline{k=0}: c_0 = \frac{1}{2\pi} \left[ \int_{-\pi}^0 -x dx + \int_0^{\pi} x dx \right] = \frac{1}{2\pi} \left[ \frac{\pi^2}{2} + \frac{\pi^2}{2} \right] = \frac{\pi^2}{2}$$

$$\underline{k \neq 0}: c_{ka} = \frac{1}{2\pi} \left[ -\frac{x}{ik} e^{-ikx} + \frac{1}{ik} e^{-ikx} \right]_{-\pi}^0 + \frac{1}{2\pi} \left[ -\frac{x}{ik} e^{-ikx} + \frac{1}{ik} e^{-ikx} \right]_0^{\pi}$$

$$= -\frac{1}{2\pi} \left[ +\frac{1}{k^2} - \frac{\pi}{ik} e^{ik\pi} - \frac{1}{k^2} e^{ik\pi} \right]$$

$$+ \frac{1}{2\pi} \left[ -\frac{\pi}{ik} e^{-ik\pi} + \frac{1}{k^2} e^{-ik\pi} - \frac{1}{k^2} \right]$$

$$= -\frac{1}{\pi k^2} + \left[ \frac{1}{2ik} + \frac{1}{2\pi k^2} \right] e^{ik\pi} + \left[ -\frac{1}{2ik} + \frac{1}{2\pi k^2} \right] e^{-ik\pi}$$

$$\tilde{FS}[f(x)] = \sum_{k=-\infty}^{\infty} c_{ka} e^{ikx}, \quad c_{ka} \text{ written above}$$

But  $e^{ik\pi} = \cos(k\pi) + i \sin(k\pi) = \cos(k\pi) = (-1)^k = e^{-ik\pi}$

 $c_{ka} = \frac{1}{\pi k^2} [1 + (-1)^k] \in \begin{cases} 0 & \text{if } k=2j \\ \frac{2}{\pi k^2} & \text{if } k=2j+1 \end{cases} \Rightarrow \tilde{FS}[f(x)] = \frac{\pi}{2} + \sum_{j=0}^{\infty} \frac{2e^{ix(2j+1)}}{\pi(2j+1)^2}$

$$e) f(x) = |\sin x| = \begin{cases} \sin x & \sin x \geq 0 \\ -\sin x & \sin x \leq 0. \end{cases}$$

$$\tilde{FS}[f(x)] = \sum_{k=-\infty}^{\infty} c_{ka} e^{ikx}$$

$$c_{ka} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \left[ \int_{-\pi}^0 -\sin x e^{-ikx} dx \right.$$

$$= \frac{1}{2\pi} \left[ -\int_{-\pi}^0 \frac{e^{ix} - e^{-ix}}{2i} e^{-ikx} dx \right] + \int_0^{\pi} \sin x e^{-ikx} dx$$

$$+ \frac{1}{2\pi} \left[ \int_0^{\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-ikx} dx \right]$$

$$= \frac{1}{4\pi i} \left[ - \int_{-\pi}^0 (e^{i(1-k)x} - e^{-i(1+k)x}) dx + \int_0^{\pi} (e^{i(1-k)x} - e^{-i(1+k)x}) dx \right]$$

$$\underline{k \neq 1 \text{ and } k \neq -1}: c_{ka} = \frac{1}{4\pi i} \left[ \frac{-1}{i(1-k)} e^{i(1-k)x} + \frac{1}{i(1+k)} e^{-i(1+k)x} \right]_{-\pi}^0$$

$$+ \frac{1}{4\pi i} \left[ \frac{1}{i(1-k)} e^{i(1-k)x} + \frac{1}{i(1+k)} e^{-i(1+k)x} \right]_0^{\pi}$$

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$$\begin{aligned}
3. \quad C_{ka} &= \frac{1}{4\pi i} \left[ \cancel{\frac{-1}{i(1-k)}} + \frac{1}{i(1+k)} + \frac{1}{i(1-k)} e^{-i\pi(1-k)} + \frac{1}{i(1+k)} e^{i\pi(1+k)} \right] \\
&\quad + \frac{1}{4\pi i} \left[ \frac{1}{i(1-k)} e^{i\pi(1-k)} + \frac{1}{i(1+k)} e^{-i\pi(1+k)} - \cancel{\frac{1}{i(1-k)}} - \cancel{\frac{1}{i(1+k)}} \right] \\
&= \frac{1}{4\pi i} \left[ \frac{1}{i(1-k)} \left( e^{i\pi(1-k)} + e^{-i\pi(1-k)} \right) + \frac{1}{i(1+k)} \left( e^{-i\pi(1+k)} + e^{i\pi(1+k)} \right) \right. \\
&\quad \left. + \frac{1}{2\pi(1-k)} + \frac{1}{2\pi(1+k)} \right] \\
&= \frac{-1}{2\pi(1-k)} \cos((1-k)\pi) - \frac{1}{2\pi(1+k)} \cos((1+k)\pi) + \frac{1}{\pi(1-k)(1+k)} \\
&= \begin{cases} 0 & , k_a \text{ odd}, k_a \neq -1, k_a \neq 1 \\ \frac{2}{\pi(1-k)(1+k)} & , k_a \text{ even}, k_a = 2j \end{cases}
\end{aligned}$$

$$\begin{aligned}
\underline{k_a=1}: \quad C_1 &= \frac{1}{4\pi i} \left[ - \int_{-\pi}^0 (1 - e^{-2ix}) dx + \int_0^\pi (1 - e^{-2ix}) dx \right] \\
&= \frac{1}{4\pi i} \left[ -x - \frac{1}{2i} e^{-2ix} \right]_{-\pi}^0 + \frac{1}{4\pi i} \left[ x + \frac{1}{2i} e^{-2ix} \right]_0^\pi \\
&= \frac{1}{4\pi i} \left[ -\cancel{\frac{1}{2i}\pi} + \cancel{\frac{1}{2i}e^{2i\pi}} \right] + \frac{1}{4\pi i} \left[ \pi + \cancel{\frac{1}{2i}e^{-2i\pi}} - \cancel{\frac{1}{2i}} \right] \\
&= \frac{-1}{4i} + \frac{1}{4i} = 0
\end{aligned}$$

$$\begin{aligned}
\underline{k_a=-1}: \quad C_{-1} &= \frac{1}{4\pi i} \left[ \frac{-1}{2i} e^{2ix} + x \right]_{-\pi}^0 + \frac{1}{4\pi i} \left[ \frac{1}{2i} e^{2ix} - x \right]_0^\pi \\
&= \frac{1}{4\pi i} \left[ -\cancel{\frac{1}{2i}} + \cancel{\frac{1}{2i}e^{-2i\pi}} + \pi \right] + \frac{1}{4\pi i} \left[ \cancel{\frac{1}{2i}e^{2i\pi}} - \pi - \cancel{\frac{1}{2i}} \right] \\
&= 0
\end{aligned}$$

$$\widetilde{f}_S(f(x)) = \frac{2}{\pi} \sum_{j=-\infty}^{\infty} \frac{e^{2jix}}{(1-4j^2)}$$

$$g) f(x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0. \end{cases}$$

$$\tilde{f}_S(f(x)) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_0^{\pi} x e^{-ikx} dx$$

$$k \neq 0: \quad = \frac{1}{2\pi} \left[ -\frac{x}{ik} e^{-ikx} + \frac{1}{k^2} e^{-ikx} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[ -\frac{\pi}{ik} e^{-ik\pi} + \frac{1}{k^2} e^{-ik\pi} - \frac{1}{k^2} \right]$$

$$= \left[ -\frac{1}{2ik} + \frac{1}{2\pi k^2} \right] e^{-ik\pi} - \frac{1}{2\pi k^2}$$

↓  
cos kπ - i sin kπ

$$= \left[ \frac{i}{2ik} + \frac{1}{2\pi k^2} \right] \cos k\pi - \frac{1}{2\pi k^2}$$

$$= \begin{cases} \frac{i}{2ik} & k \text{ even}, k=2j \\ -\frac{i}{2ik} - \frac{1}{\pi k^2} & k \text{ odd}, k=2j+1 \end{cases}$$

$$k=0: \quad c_0 = \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \cdot \frac{\pi^2}{2} = \frac{\pi}{4}.$$

$$\tilde{f}_S(f(x)) = \frac{\pi}{4} + \sum_{j=-\infty, j \neq 0}^{\infty} \frac{i}{4j} e^{-i(2j)x} - \sum_{j=-\infty}^{\infty} \left[ \frac{i}{2(2j+1)} + \frac{1}{\pi(2j+1)^2} e^{-i(2j+1)x} \right]$$

3.2.53-  $a \in \mathbb{R}$ ,  $f(x) = e^{ax}$

$$\text{FS}[e^{ax}] = \frac{\sinh a\pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k (a+ik)}{a^2+k^2} e^{ikx}$$

$$u_k(x) = (a+ik) e^{ikx}$$

$$u_{-k}(x) = (a-ik) e^{-ikx}$$

$$u_k(x) + u_{-k}(x) = 2a \cos kx - 2ik \sin kx.$$

$$\begin{aligned} \text{FS}[e^{ax}] &= \frac{\sinh a\pi}{\pi} \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{a^2+k^2} (u_k(x) + u_{-k}(x)) \right) \\ &\approx \frac{\sinh a\pi}{\pi} + 2 \frac{\sinh a\pi}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{a^2+k^2} (a \cos kx - ik \sin kx) \end{aligned}$$

b)  $f(x) = e^{ax}$

$$\text{FS}[e^{ax}] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx = \left[ \frac{1}{\pi a} e^{ax} \right]_{-\pi}^{\pi} = \frac{1}{\pi a} [e^{a\pi} - e^{-a\pi}]$$

$$= \frac{2}{\pi a} \sinh a\pi$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos kx dx = \frac{1}{\pi} \left[ \frac{e^{ax}}{a^2+k^2} (a \cos kx + k \sin kx) \right]_{-\pi}^{\pi}$$

integ by parts (check assignment 4)

$$= \frac{1}{\pi} \left[ \frac{e^{a\pi}}{a^2+k^2} (a \cos a\pi) - \frac{e^{-a\pi}}{a^2+k^2} (a \cos a\pi) \right]$$

$$= \frac{a \cos a\pi}{\pi} \left[ \frac{e^{a\pi} - e^{-a\pi}}{a^2+k^2} \right] = \frac{2a}{\pi} \frac{\cos a\pi \sinh a\pi}{a^2+k^2}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin kx dx = \frac{1}{\pi} \left[ \frac{e^{ax}}{a^2+k^2} (a \sin kx - k \cos kx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[ \frac{e^{a\pi}}{a^2+k^2} (-k \cos a\pi) - \frac{e^{-a\pi}}{a^2+k^2} (-k \cos a\pi) \right]$$

$$= \frac{1}{\pi} \cdot \frac{-k}{a^2+k^2} \cdot \cos a\pi \cdot 2 \sinh a\pi$$

$$= \frac{2ka(-1)^{k+1}}{\pi(a^2+k^2)} \sinh a\pi$$

$$\begin{aligned} \text{FS}[e^{ax}] &= \frac{a_0}{2} \sinh a\pi + \sum_{k=1}^{\infty} \frac{2a(-1)^k}{\pi(a^2+k^2)} \sinh a\pi \cdot \frac{1}{a^2+k^2} \cdot \cos kx \\ &\quad + \sum_{k=1}^{\infty} \frac{2ka(-1)^k}{\pi(a^2+k^2)} \sinh a\pi \sin kx \end{aligned}$$

$$6 = \frac{1}{\pi a} \sinh ax + \frac{2}{\pi} \sinh ax \sum_{k=1}^{\infty} \frac{a(-1)^k}{a^2+k^2} \cos kx + \frac{ka(-1)^k}{a^2+k^2} \sin kx$$

$$\frac{1}{\pi} \sinh ax = \frac{\sinh ax}{\pi} \left[ \frac{1}{a} + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{a^2+k^2} (\cos kx + k \sin kx) \right]$$

$$3.2.55-a) \quad \widetilde{FS}[xe^{ix}] = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{ix} e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{i(x-k)} dx$$

$$\underline{k=1}: \quad c_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underset{\text{odd}}{x} dx = 0$$

$$\underline{k \neq 1}: \quad c_k = \frac{1}{2\pi} \left[ \frac{x}{i(1-k)} e^{i(x-k)} + \frac{1}{(1-k)^2} e^{i(x-k)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[ \frac{\pi}{i(1-k)} e^{i\pi(1-k)} + \frac{1}{(1-k)^2} e^{i\pi(1-k)} \right]$$

$$+ \frac{1}{2\pi} \left[ \frac{\pi}{i(1-k)} e^{-i\pi(1-k)} - \frac{1}{(1-k)^2} e^{-i\pi(1-k)} \right]$$

$$= \frac{1}{2\pi} \left[ \frac{\pi}{i(1-k)} \cdot 2\cos((1-k)\pi) + \frac{1}{(1-k)^2} \cdot 2i \sin((1-k)\pi) \right]$$

$$= \frac{-i}{1-k} \cos((1-k)\pi) = \frac{-i}{1-k} (-1)^{k+1} 0$$

$$\widetilde{FS}[xe^{ix}] = \frac{(-1)^{k+1} i}{1-k} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k} i e^{ikx}$$

$$\begin{array}{c} x \\ | \sqrt{x} \\ 0 \end{array} \quad \begin{array}{c} e^{ix(1-k)} \\ \frac{1}{i(1-k)} e^{ix(1-k)} \\ \frac{-1}{(1-k)^2} e^{ix(1-k)} \end{array}$$

$$b) \tilde{FS}[xe^{ix}] = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k} i (\cos kx + i \sin kx)$$

$$\therefore \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k} \cos kx - \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k} \sin kx$$

$$\tilde{FS}[x \cos x] = \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}}{1-k} \sin kx$$

$$\tilde{FS}[x \sin x] = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k} \cos kx$$

3.2.57- True

$$\tilde{FS}[g(x) + i h(x)] = FS[g(x)] + i FS[h(x)]$$

$$\tilde{FS}[f(x)] = \tilde{FS}[g(x)] + i FS[h(x)]$$

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=1}^{\infty} g_k e^{ikx} + i \sum_{k=1}^{\infty} h_k e^{ikx}$$

$$\therefore \tilde{FS}[f(x)] = \sum_{k=1}^{\infty} (g_k + ih_k) e^{ikx}$$

$\therefore g_k = \text{Re } c_k$

$h_k = \text{Im } c_k$

$$3.2.59-a) \tilde{FS}[f(x)] = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

$$\tilde{FS}[f(x)e^{ix}] = \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{ikx}, \tilde{c}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ix} e^{-ikx} dx$$

We have to show that  $\tilde{c}_k = c_{k-1}$

$$\tilde{c}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ix} e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i(k-1)x} dx = c_{k-1}$$

$$b) \tilde{c}_k = c_{k+m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i(k+m)x} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} e^{-ihx} dx$$

$f(x) e^{-imx}$  has complex Fourier coeff  $\hat{c}_n = c_{k+m}$

$$\text{c)} \quad C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$$

$$\begin{aligned} C_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \\ &= \frac{1}{2} [a_k - i b_k] \end{aligned}$$

by a)  $f(x)e^{ix}$  has Fourier coeff  $\tilde{C}_k = C_{k-1}$

$$\tilde{C}_k = \frac{1}{2} [a_{k-1} - i b_{k-1}]$$

$$\text{FS}[f(x)e^{ix}] = \sum_{k=-\infty}^{\infty} \tilde{C}_k e^{ikx}$$

$$\text{FS}[f(x)\cos kx + i f(x)\sin kx] = \sum_{k=-\infty}^{\infty} \frac{1}{2} (a_{k-1} - i b_{k-1}) (\cos kx + i \sin kx)$$

$$\text{FS}[f(x)\cos kx] + i \text{FS}[f(x)\sin kx] = \sum_{k=-\infty}^{\infty} \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx$$

$$+ i \sum_{k=-\infty}^{\infty} \frac{1}{2} a_{k-1} \sin kx - \frac{1}{2} b_{k-1} \cos kx$$

$$\text{FS}[f(x)\cos kx] = \sum_{k=-\infty}^{\infty} \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx$$

$$= \sum_{k=-\infty}^{-2} \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx$$

A  $a_{-k} = a_k$   
 $b_{-k} = -b_k$

$$+ \sum_{k=2}^{\infty} \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx$$

$$+ \frac{1}{2} a_{-1} + \frac{1}{2} a_0 \cos x + \frac{1}{2} b_0 \sin x + \frac{1}{2} a_{-2} \cos(-x)$$

$$+ \frac{1}{2} b_{-2} \sin(-x)$$

$$= \sum_{k=2}^{\infty} \left( \frac{1}{2} a_{-k-1} \cos(-kx) + \frac{1}{2} b_{-k-1} \sin(-kx) \right)$$

$$+ \sum_{k=2}^{\infty} \left( \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx \right) + \frac{1}{2} a_1 + \frac{1}{2} a_0 \cos x$$

$$+ \frac{1}{2} b_0 \sin x + \frac{1}{2} a_2 \cos x + \frac{1}{2} b_2 \sin x$$

$$= \sum_{k=2}^{\infty} \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx$$

$$+ \sum_{k=2}^{\infty} \frac{1}{2} a_{k-1} \cos kx + \frac{1}{2} b_{k-1} \sin kx$$

$$+ \frac{1}{2} a_0 \cos x + \frac{1}{2} b_0 \sin x + \frac{1}{2} a_1 + \frac{1}{2} a_2 \cos x + \frac{1}{2} b_2 \sin x$$

$$= \frac{1}{2}a_1 + \sum_{k=1}^{\infty} \frac{1}{2}a_{k+1} \cos kx + \frac{1}{2}b_{k+1} \sin kx \\ + \sum_{k=1}^{\infty} \frac{1}{2}a_{k-1} \cos kx + \frac{1}{2}b_{k-1} \sin kx$$

$$\underline{b_0 = 0} \quad = \frac{1}{2}a_1 + \frac{1}{2} \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) \cos kx + (b_{k+1} + b_{k-1}) \sin kx$$

$$\tilde{FS}[f(x) \sin x] = \sum_{k=-\infty}^{\infty} \frac{1}{2}a_{k+1} \sin kx - \frac{1}{2}b_{k-1} \cos kx \\ = -\frac{1}{2}b_1 + \sum_{k=-\infty}^{\infty} \frac{1}{2}a_{k-1} \sin kx - \frac{1}{2}b_{k-1} \cos kx \\ + \sum_{k=-\infty}^{-1} \frac{1}{2}a_{k+1} \sin kx - \frac{1}{2}b_{k+1} \cos kx \\ = \sum_{k=1}^{\infty} \frac{1}{2}a_{k-1} \sin kx - \frac{1}{2}b_{k-1} \cos kx \\ + \sum_{k=1}^{\infty} -\frac{1}{2}a_{k+1} \sin kx + \frac{1}{2}b_{k+1} \cos kx + \frac{1}{2}b_1 \\ = \frac{1}{2} \sum_{k=1}^{\infty} (a_{k-1} - a_{k+1}) \sin kx + (b_{k+1} - b_{k-1}) \cos kx \\ + \frac{1}{2}b_1$$

$$3.3.1 \text{ a) } \tilde{F}_3[f(x)] = \frac{1}{2} + \frac{2}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \dots \right) \quad f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\text{b) } \tilde{F}_5[f_5(x) - \frac{\pi}{4}x] = m + \frac{2}{\pi} \left( -\cos x - \frac{1}{9} \cos 3x - \dots \right)$$

$$m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = \frac{\pi}{4}$$

$$\tilde{F}_5[f_5(x)] = \frac{\pi}{4} + \frac{2}{\pi} \left( -\cos x - \frac{1}{9} \cos 3x - \dots \right) + \frac{1}{2} \tilde{F}_5(x)$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k}$$

$$\text{b) } \tilde{F}_5[f_5(x) - \frac{\pi}{4}x] = m + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k+1)x)}{(2k+1)^3} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos kx$$

$$m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_5(x) dx = \frac{1}{2\pi} \int_0^{\pi} \frac{1}{2} x^2 dx = \frac{1}{12\pi} [x^3]_0^{\pi} = \frac{\pi^2}{12}$$

$$\tilde{F}_5[f_5(x)] = \frac{\pi^2}{12} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{(2k+1)^3} + \frac{\pi}{4} \tilde{F}_5(x)$$

$$= \frac{\pi^2}{12} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{(2k+1)^3} + \frac{\pi}{4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k}$$

3.2.2.

$$\text{a) } f(x) = x^3 \quad \tilde{F}[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 dx = 0$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \cos kx dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin kx dx$$

$$= \frac{1}{\pi} \left[ -\frac{x^3}{k^2} \cos kx + \frac{3x^2}{k^2} \sin kx + \frac{6x}{k^3} \cos kx - \frac{6}{k^4} \sin kx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[ -\frac{\pi^3}{k^2} \cos k\pi + \frac{6\pi^2}{k^3} \sin k\pi - \frac{\pi^3}{k^2} \cos k\pi + \frac{6\pi}{k^3} \cos k\pi \right]$$

$$= \frac{1}{\pi k^3} (-\pi^3 k^2 + 6\pi^2 - \pi^3 k^2 + 6\pi) \cos k\pi$$

$$= \frac{12 - 2\pi^2 k^2}{k^3} (-1)^k$$

$$\begin{aligned} x^3 &\rightarrow \sin kx \\ 3x^2 &\rightarrow -\frac{1}{k} \cos kx \\ 6x &\rightarrow -\frac{1}{k^2} \sin kx \\ 6 &\rightarrow \frac{1}{k^3} \cos kx \\ 0 &\rightarrow \frac{1}{k^4} \sin kx \end{aligned}$$

$$\tilde{FS}[x^3] = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} (12 - 2\pi^2 k^2) \sin kx$$

$$[\tilde{FS}[x^3]]' = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} (12 - 2\pi^2 k^2) \cdot k \cos kx$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} (12 - 2\pi^2 k^2) \cos kx$$

$$\tilde{FS}[x^2] = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 4}{k^2} \cos kx \quad (3.2.1)$$

They are  $\neq$  since the  $2\pi$ -periodic Fourier extension of  $x^3$  is not continuous (Thm 3.22)

$$3.3.3. f(x) = x^4$$

$$\tilde{FS}[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$b_k = 0 \quad (f \text{ is even})$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \left[ \frac{x^5}{5\pi} \right]_{-\pi}^{\pi} = \frac{2\pi^5}{5\pi} = \frac{2}{5}\pi^4$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos kx dx = \frac{2}{\pi} \int_0^{\pi} x^4 \cos kx dx$$

$$= \frac{2}{\pi} \left[ \frac{x^4}{k^4} \sin kx + \frac{4x^3}{k^2} \cos kx \right]_0^{\pi}$$

$$+ \frac{2}{\pi} \left[ -\frac{12x^2}{k^3} \sin kx - \frac{24x}{k^5} \cos kx + \frac{24}{k^7} \sin kx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[ \frac{4\pi^3}{k^2} (-1)^k - \frac{24\pi}{k^5} (-1)^k \right]$$

$$= \frac{2}{\pi} \frac{(-1)^k}{k^4} \left[ 4\pi^3 k^2 - 24\pi \right]$$

$$= \frac{2(-1)^k}{k^4} (4\pi^2 k^2 - 24)$$

$$\tilde{FS}[x^4] = \frac{1}{5}\pi^4 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{k^4} (4\pi^2 k^2 - 24) \cos kx$$

$$[\tilde{FS}[x^4]]' = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k^3} (4\pi^2 k^2 - 24) \sin kx$$

$$4[\tilde{FS}[x^3]] = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} (48 - 8\pi^2 k^2) \sin kx = [\tilde{FS}[x^4]]'$$

$$3.3.4 - \tilde{FS}[x^2] = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^2} \cos kx \quad (3.2.1) \quad 11$$

5  $\tilde{FS}\left[\frac{x^3}{3} - \frac{\pi^2}{3}x\right] = m + \sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^3} \sin kx$

$$m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^3}{3} dx = \left[ \frac{1}{24\pi} x^4 \right]_{-\pi}^{\pi} = 0$$

$$\begin{aligned} \tilde{FS}[x^3] &= 3 \sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^3} \sin kx + FS[\pi^2 x] \\ &= \tilde{FS}\left[\frac{x^4}{4}\right] = -3 \sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^4} \cos kx + 2\pi^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k \pi}{k^4} \sin kx \end{aligned}$$

$$m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^4}{4} dx = \frac{1}{40\pi} [2\pi^5] = \frac{\pi^4}{20}$$

$$\tilde{FS}[x^4] = \frac{\pi^4}{5} - 4 \left[ \sum_{k=1}^{\infty} \left( \frac{(-1)^{k+1} 2}{k^4} + \frac{2\pi^2 (-1)^{k-1}}{k^4} \right) \cos kx \right]$$

3.3.5 -  $x=0$ :  $0 = \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}$$

$$x = \frac{\pi}{2} : \frac{1}{2} \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos \frac{k\pi}{2}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^{2j-1} (-1)^j}{4j^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos \frac{k\pi}{2} = \left( -\frac{\pi^2}{8} + \frac{\pi^2}{6} \right) \cdot \frac{1}{2}$$

$$= \frac{\pi^2}{12} - \frac{\pi^2}{16}$$

6  $x = \frac{\pi}{3} : \frac{1}{2} \left(\frac{\pi}{3}\right)^2 = \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos \frac{k\pi}{3}$

$$\cos \frac{k\pi}{2} = \begin{cases} 0 & k=2j+1 \\ (-1)^j & k=2j \end{cases}$$

$$\frac{\pi^2}{18} = \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos \frac{k\pi}{3}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos \frac{k\pi}{3} = \frac{1}{2} \left( \frac{\pi^2}{6} - \frac{\pi^2}{18} \right)$$